Fundamental Solution

Note Title 1/15/2010

We want to find a formula for the solution of

(1) $\quad Lf = g, \; f(0)=0, \; f'(0)=0, \; \dots, \; f^{(n-1)}(0)=0,$

where $L = D^n + a_{n-1}D^{n-1} + \dots + a_1 D + a_0 I$, and

$D = \frac{d}{dx}$.

We assume $g(x)$ is continuous on $\mathbb{R}$.

Let w be the unique solution of

$$Lw = 0, \; w(0)=0, \; \dots, \; w^{(n-2)}(0)=0, \; w^{(n-1)}(0)=1.$$

This solution is described in primer on differential equations.

Let

(2) $$f(x) = \int_0^x w(x-y)\,g(y)\,dy.$$

Then $f(0)=0$.

$$f'(x) = \underbrace{w(0)}_{=0}\,g(x) + \int_0^x w'(x-y)\,g(y)\,dy,$$

$$f''(x) = \underbrace{w'(0)}_{=0}\,g(x) + \int_0^x w''(x-y)\,g(y)\,dy,$$

$$\vdots$$

$$f^{(n)}(x) = \underbrace{w^{(n-1)}(0)}_{1}\,g(x) + \int_0^x w^{(n)}(x-y)\,g(y)\,dy.$$

Hence

$$f(0) = 0,\ f'(0) = 0,\ \dots,\ f^{(n-1)}(0) = 0.$$

Also

$$Lf(x) = \int_0^x \underbrace{Lw(x-y)}_{=0}\, g(y)\, dy + g(x)$$

$$= g(x),$$

which is (1).